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# A statistical model for collective instabilities

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**Abstract.** We introduce a simple stochastic model for the collective evolution of a population of elements which can assume only a finite number $n$ of states ('damage') $d=k/n$ for $0 \leqslant k<n$. The evolution is given by a probabilistic rule which depends only on a function of the damage, and diverges for $d=1$. We consider mostly cases where a homogeneous evolution (same damage for all elements) is unstable. Our aim is to characterize the final state of the system (i.e. the statistical distribution of the damage) in the thermodynamic limit. A non-trivial scaling with the number of damage states is observed. The scaling variable $(1-d)n^{\beta}$ accounts for the $n$-dependence of most properties of the model. The exponent $\beta$ is only a function of the singularity of the probability, and its expression is obtained from a mapping to a convection–diffusion problem. The dependence on the number of elements leads to logarithmic corrections which are discussed.

## 1. Introduction

The aim of this article is to propose a model suitable for describing the statistics of a set of elements whose collective evolution is unstable. For this purpose, we propose a simple 'mean-field' *stochastic* model. The probabilistic nature of the evolution is an essential feature. The competition between the average driving force of the instability and the statistical fluctuations leads to non-trivial scaling laws.

Typical applications are stochastic growth models [1], some hydrodynamic instabilities such as viscous fingering [12], or fracture models [3]. However, the model is not restricted to those examples and it constitutes a general template which can easily be tailored to other physical problems. We will come back more precisely on these applications after having defined the model.

## 2. Model

We consider a population of $m$ elements. Each one is characterized by a 'damage', $d$, which can assume only $n$ different values: $d=k/n$ for $0 \leqslant k \leqslant (n-1)$. Moreover, the damage is irreversible, and can only increase. Initially, all elements have zero damage. The evolution is given by the following probabilistic rule: At each time step,

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we randomly choose a site $i$ with a probability $p_i$ proportional to a given function of its damage $\varphi(d_i)$. Normalization of the probability leads to the expression

$$p_i = \varphi(d_i) \Big/ \sum_{j=1}^{m} \varphi(d_j). \tag{1}$$

The damage at this site is increased to $d_i' = d_i + 1/n$. The process is repeated until one site reaches $d = 1$. (The state $d = 1$ is not counted as a state since it is never occupied during the process, and as soon as it is, the evolution stops!)

A central element of the model is the function $\varphi(d)$ which characterizes the stability of the system. If $\varphi(x)$ was a constant, all elements would be damaged with the same probability, and thus our model reduces in this limit to a mere diffusion process. When $\varphi(d)$ is an increasing function of the damage, the 'homogeneous' evolution of the system is unstable. Indeed an element which has a larger damage than the other will tend to be more easily damaged. This is the situation we are mostly interested in. We will consider functions with a singular behaviour for $d = 1$:

$$\varphi(d) = \frac{1}{(1-d)^{\alpha}}. \tag{2}$$

The divergence of $\varphi$ for $d = 1$ legitimizes our stopping criterion. Indeed, if an element $i$ would reach this limit value, the probability of further increase of this site would be 1, and that of the other elements 0. Thus the system would be quenched at this stage of evolution. In this study, we focus on the scaling properties of the problem, and thus, we expect that these are solely dependent on the singularity (i.e. $\alpha$) of $\varphi(d)$ at the point where it diverges.

Variants of this model could be considered, such as the stable case where $\varphi(d)$ is a decreasing function of the damage. In such a case, the evolution is stable since a less damaged element will have a larger probability of being damaged. We will see below that there exists a small region of stability (namely $-\frac{1}{2} < \alpha < 0$) where non-trivial scaling properties are encountered. Another case of physical interest would be the situation where $\varphi$ increases continuously without diverging for a finite value of the damage (e.g. $\varphi(d) = d^{\alpha}$).

Let us note finally a few points before discussing the results. The coupling between different elements is reduced to a very simple form. It appears only in the expression of the probability of evolution, due to the normalization condition (1). Since there is no geometrical coupling between elements, the model can be considered as a mean-field description, in as much as any element interacts with any other equally. Elements should here be considered in a broad sense. They can be material elements (see below), but also Fourier modes for instance.

For applications to the field of growth models, such as the celebrated diffusion limited aggregation model [4], or variants of it such as the dielectric breakdown model [5], we can consider 'elements' to be the tips of the growing cluster or propagating crack, respectively. In a finite geometry such as most often considered experimentally, the process will stop as soon as one tip has reached the boundary. If we assume that the rate of growth of a tip is simply a function of its distance to this boundary—or $(1-d)$ in the language we use to define the model—we are dealing with our precise model. The growth probability does not take into account the real tip interactions, but rather some statistical 'screening' does occur because of the normalization condition we imposed.

For viscous fingering [2] we can use the same picture due to the well-known correspondence with the previous model. It is interesting to note that by introducing a non-Newtonian fluid, such as some polymer solutions which display a power-law behaviour between the effective viscosity and the shear rate, one can easily vary the weight function $\varphi$, and in particular its singularity $\alpha$.

Finally, in the case of fracture, one can consider an 'element' as a slice of material. The damage $d$ for this slice can be considered to be the usual 'damage' (i.e. relative reduction of elastic modulus due to the microcracks in the element). An assumption of 'locality' which would relate the probability of further damage to the mean stress in the element—and thus to its damage for a homogeneous loading—leads again to the model we introduced above. This mean-field limit, may help in understanding some recently proposed scaling laws obtained from numerical simulation [6].

## 3. Final damage scaling for two elements ($m=2$)

We studied the problem using two methods, depending on the number $m$ of elements. When $m$ is small, it is convenient to deal directly with the probability functions, whereas for large $m$, we resort to a Monte-Carlo simulation. Let us first present the first case, for $m=2$.

The system is characterized at each step of the evolution by two damages $(d_i, d_2)$. The total number of damage updates is the sum of the two damages times $n$. We can thus introduce a 'time' $t=(d_1+d_2)/2$ as the average damage.

In the $(d_1, d_2)$ plane, the mean flow can be easily constructed from the $\varphi(d)$ function. At each point, the mean drift is parallel to the vector $\varphi(d_1)$, $\varphi(d_2)$). The flow lines are given by the equation $\mathrm{d}y/\mathrm{d}x=\varphi(y)/\varphi(x)$. Introducing the integral $\psi(x)=\int \mathrm{d}x/\varphi(x)$, the flow lines are given by

$$\psi(d_1)=\psi(d_2)+K \tag{3}$$

where $K$ is a constant. In the case $\alpha=1$, the flow lines are hyperbolas, $(1-d_1)^2-(1-d_2)^2=K$. Figure 1 shows the flow lines in the square $[0, 1]^2$. The divergence of the

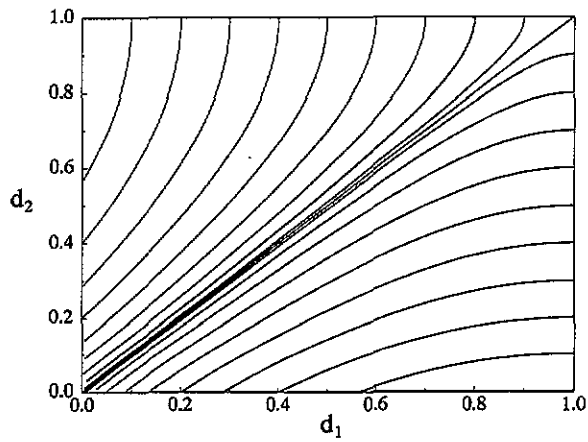


**Figure 1.** Flow lines in the $(d_1, d_2)$ plane, for $\alpha=1$. The divergence of the flow is the reason for the instability of the homogeneous evolution. In this particular case, the flow lines are hyperbolas.

flow reflects the instability of the model. For $\alpha=0$, flow lines are parallel straight lines $d_1=d_2+K$. For negative $\alpha$ (stable case), the flow lines converge toward the $(1,1)$ corner.

The problem may be seen as a convection-diffusion process, where diffusion is given by the probabilistic nature of the model, and convection is the result of the weighting function $\varphi(x)$.

At each time step, we can relate simply the probabilities $\pi(d_1, d_2)=\pi(d_1, 2(t-d_1))$ to those computed at the previous time step, through

$$\pi(d_1, d_2)=\frac{\varphi(d_1-1/n)}{\varphi(d_1-1/n)+\varphi(d_2)}\pi(d_1-1/n, d_2)+\frac{\varphi(d_2-1/n)}{\varphi(d_1)+\varphi(d_2-1/n)}\pi(d_1, d_2-1/n). \quad (4)$$

The latter equation, which is simply a rewriting of (1), leads to a simple computation of the probability field, and thus by recording the probabilities $\pi(d,1)$ or $\pi(1,d)$ (being equal), we obtain the probability $p_i$ that the damage in the surviving element at the final stage of the process, is equal to $d=i/n$: $p_i=2\pi(i/n, 1)$. We will be mostly interested in the continuum limit where the number of damage states $n$ goes to infinity. In this continuum limit, the discrete set of probabilities is more conveniently described as a probability *measure*, and hence we have to take into account the vanishing of the differential element. To this end, we introduce the functions $p_n(d)=np_i$, so that $p_n(d)\mathrm{d}d$ gives the probability that the damage lies in the interval $[d, d+\mathrm{d}d]$.

Figure 2($a$) gives an example of the functions $p_n(d)$ for a system containing two elements, a parameter $\alpha=1$, and a number of states ranging from $n=50$ to 800. We see on the figure that the number of states drastically changes the shape of the distribution. The first result we report here is that it is possible to account for this evolution through the definition of a reduced variable:

$$\delta=(1-d)n^{\beta} \quad (5)$$

where $\beta$ is a scaling exponent. If the damage appears only through the variable $\delta$, then the normalization of the probability $\int_0^1 p_n(d)\mathrm{d}d=1$ imposes the normalization factor

$$p_n(d)=n^{\beta}F(\delta). \quad (6)$$

We show on Figure 2($b$), the plot of $p_n(d)n^{-\beta}$ versus $\delta$, for the best value of $\beta=0.25$ determined for the data shown in Figure 2($a$). The data collapse is excellent, and leads to a very precise determination of $\beta$.

How general is this scaling form? Figures 3($a$) and 3($b$) show two other examples of similar rescaling obtained for other values of $\alpha$ ($\alpha=0.5$ and 2, respectively). The scaling exponent $\beta$ obtained for those cases depends on $\alpha$. For $\alpha=0.5$ the best collapse is obtained for $\beta=0.33$, whereas for $\alpha=2$, $\beta=0.17$.

In addition, let us consider the particular case of neutrality, $\alpha=0$, where $\varphi(d)$ is a constant. In this case, there is no bias, and the problem reduces to a simple random walk. In this case, the reduced variable $\delta$ again leads to a scale invariant form with a scaling exponent $\beta=\frac{1}{2}$.

The form of the scaling variable can be understood using the following argument. In the very early stage of the process, the divergence of the flow will not be sensitive. The evolution will essentially be diffusive, then as the mean damage increase, the convection will be dominant. The crossover point $d^*$ between the effect of diffusion and convection is given by the local bias in the damage evolution probability, and thus it is function of the damage only (independent of $n$).

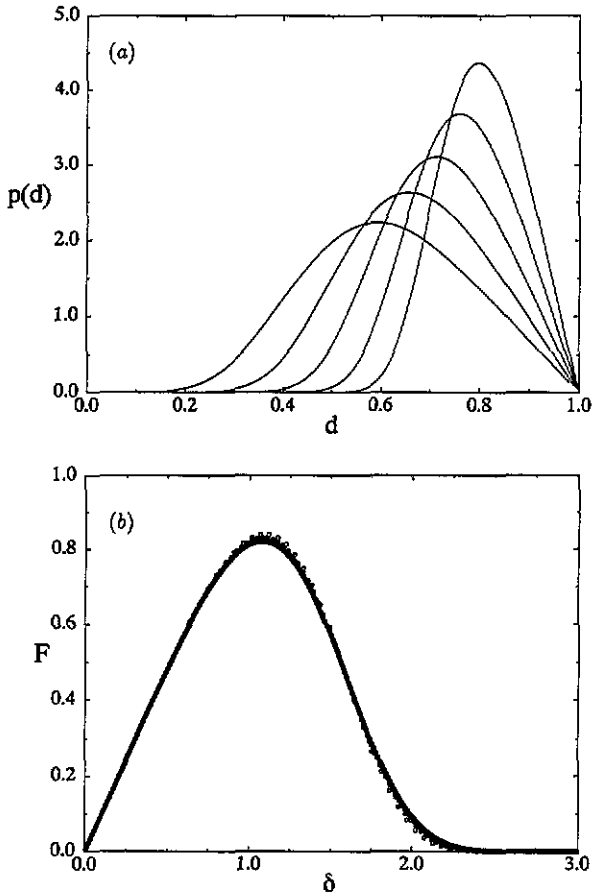


**Figure 2.** (*a*) Distribution of damage $p_n(d)$ obtained for a system of two sites ($m=2$), and $\varphi(d)=(1-d)^{-1}$. The number of damage states $n$ takes on the following values for distributions from left to right $n=50$, 100, 200, 400 and 800. (*b*) Rescaled form of the distributions of figure 2(*a*). $F=p_n(d)n^{-\beta}$ is plotted as a function of $\delta=(1-d)n^{\beta}$ for $\beta=0.25$.

In the first regime, the standard deviation of damage distribution will grow as the square root of the mean damage, and of the diffusion constant. The latter will depend on the size of the discrete damage steps performed. More precisely the diffusion constant will vary as $1/n$. Thus when the average damage is at the crossover point, the deviation will vary as $\sqrt{d^*/n}$. This difference in damage will then be mostly convected by the flow lines as sketched in figure 1. We now use (3) to obtain the expression of the flow lines: $(1-d_1)^{\alpha+1}=(1-d_2)^{\alpha+1}+K$. At the crossover time, the difference in the damage corresponds to a constant $K\propto\sqrt{d^*/n}$ and thus the final damage for the least damaged site is $(1-d)=K^{1/(1+\alpha)}\propto(n/d^*)^{-\beta}$ with

$$\beta=\frac{1}{2(1+\alpha)}. \tag{7}$$

Since $d^*$ is constant, we recover the scale invariant form $\delta=(1-d)n^{\beta}$.

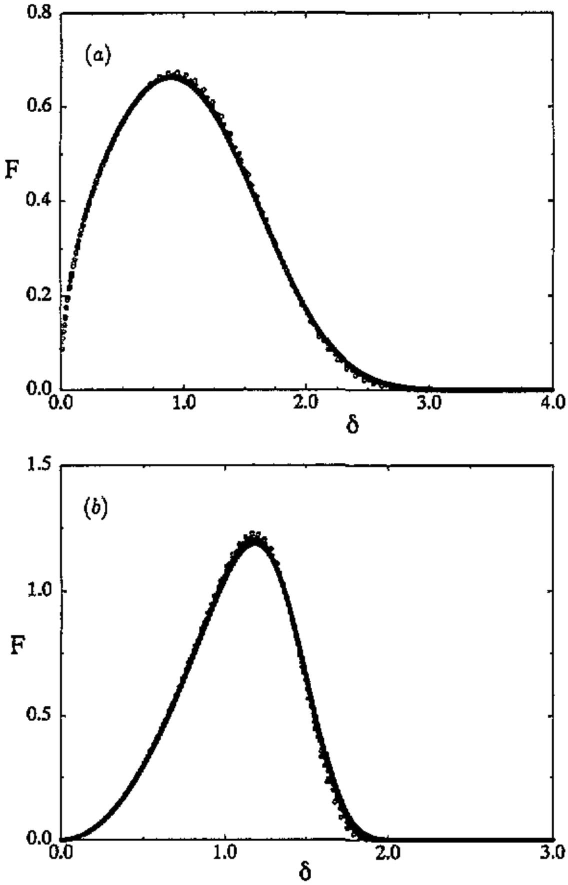


**Figure 3.** (*a*) Rescaled form of the distributions $p_n(d)$ obtained for $\alpha=0.5$. $F=p_n(d)n^{-\beta}$ is plotted as a function of $\delta=(1-d)n^{\beta}$ for $\beta=0.33$. (*b*) Rescaled form of the distributions $p_n(d)$ obtained for $\alpha=2$. $F=p_n(d)n^{-\beta}$ is plotted as a function of $\delta=(1-d)n^{\beta}$ for $\beta=0.17$.

The value of $\beta$ given by (7) is very precisely obeyed in the previous examples mentioned.

However, equation (7) cannot be valid for all values of $\alpha$. In particular $\beta$ cannot exceed 1, hence limiting (7) to $\alpha>-\frac{1}{2}$. The stable regime $-\frac{1}{2}<\alpha<0$ also displays some interesting scaling behaviour. We checked numerically an intermediate case $\alpha=-0.25$ and indeed, we obtained very good data collapse for 0.65, again very close to (7), which gives $\beta=\frac{2}{3}$. For $\alpha=-1$, we observed that the best data collapse is obtained for $\beta=1$. The relation between $\beta$ and $\alpha$ is sketched on figure 4.

The function $F$ permits an $n$-independent characterization of the probability distribution of the damage at the final stage of the process. However, $F$ does depend on the function $\varphi(d)$, and thus in the cases we study here, on $\alpha$. The general shape of the function nevertheless shares some common features for various $\alpha$ (see figure 5). For small values of $\delta$, i.e. for a damage close to 1, $F(\beta)$ behaves as a power law with an exponent equal to $\alpha$. Again this property can be explained by the same argument as the one which leads to the value $\beta$. The Gaussian distribution of damage—which was the result of the diffusion process prior to the crossover damage $d^*$—has a constant density at the apex $(d_1=d_2)$ up to second-order terms in the deviation

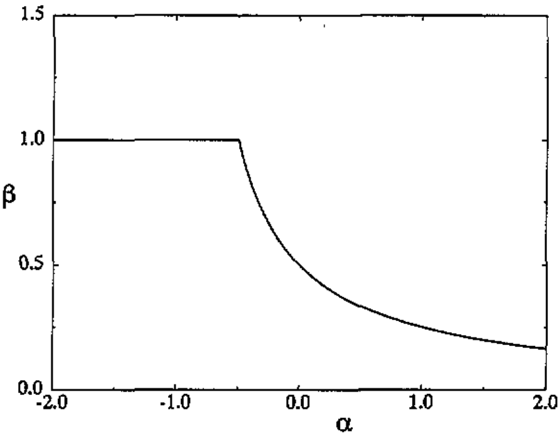


**Figure 4.** Relation between the scaling exponent $\beta$ and the singularity $\alpha$ of $\varphi(d)$.

$((d_1-d_2)\ll\sqrt{d^*/n})$. This small region will be stretched by the convection field and a constant probability at the crossover damage will give rise to a $(1-d)^\alpha$ behaviour close to 1, at the final stage. The upper limit of the power-law regime is a very abrupt cut-off for $\delta=\delta_0$. This corresponds to the fast decay of the Gaussian at the crossover point which is merely convected to the edge $d=1$ by the convection field.

This leads to the following property: If we are interested in the expectation value of $\delta^{-a}$ where $a$ is positive parameter, then two cases are to be distinguished: either $a$ is smaller than $\alpha+1$, and thus the expectation value of $\delta^{-a}$ is controlled by the apex of the $F$ function $\langle\delta^{-a}\rangle\propto\delta_0^{-a}$, or $a$ is larger than $\alpha+1$ and the expectation value $\langle\delta^{-a}\rangle\propto\int_0^{\delta_0}x^{-a}x^{\alpha}\,dx$ diverges (i.e. it is controlled by the most damaged surviving element). In particular, the expectation value of $\varphi(d)$ corresponds exactly to the case $a=\alpha$ and thus $\langle\varphi(d)\rangle$ is not influenced by the most damaged element, but it is determined by the most probable damage—we will come back on this property later.

It is also interesting to compute the probability of increase the damage of an element having a damage $d$. This probability is proportional to the number of

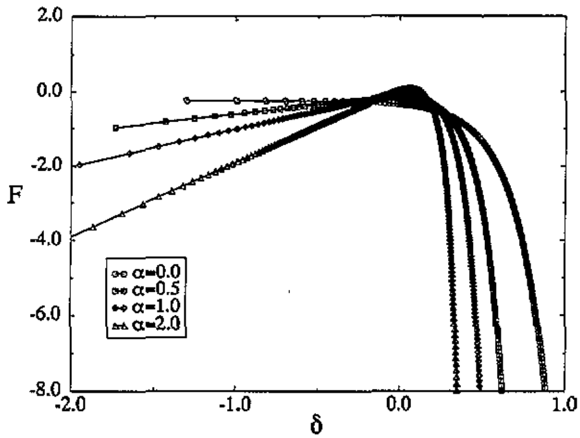


**Figure 5.** The distribution $F(\delta)$ for $\alpha=1.0$ shown in log–log coordinates with the power-law fit to the small $\delta$ behaviour with exponent $\alpha$. Similar behaviours are observed for all values of $\alpha$.

elements having this damage times the weight $\varphi(d)$. The behaviour of $F(d)$ close to $d=1$ shows that this probability is independent of the damage: it is constant for $0<\delta<\delta_0$. Moreover, all values of the exponent $\alpha$ give this constant probability property.

## 4. Final damage scaling for a large number of elements

Let us now turn to the case where the number of elements is large. The previous technique would require one to deal with the probability distribution $\pi(d_1, d_2, \ldots, d_m)$, which becomes quickly prohibitive. In this case, we used a Monte-Carlo technique. More precisely, we considered the histogram of the damage states, i.e. the number $N_i$ of elements having a damage $d=i/n$. At each time step, the probability of selecting an element with $d=i/n$ is equal to $N_i\varphi(i/n)/(\Sigma_j N_j\varphi(j/n))$. With such a probability we update the histogram $N_i \to N_i - 1$ and $N_{i+1} \to N_{i+1}+1$. The process is repeated until $N_n=1$. At the end of the process, we record the distribution of the damage, as in the previous case, to construct a probability distribution $p_{n,m}(i/n)=\langle nN_i/(m-1)\rangle$ where the angular brackets stand for an average over realizations.

The previous scaling variable $\delta=(1-d)n^\beta$ leads again to a $n$-independent distribution,

$$p_{n,m}(d)n^{-\beta}=F_m((1-d)n^\beta)=F_m(\delta) \tag{8}$$

with $\beta$ equal to the value obtained for the same $\varphi(d)$ function and only $m=2$ elements. Figure 6 shows two examples of data collapse obtained for two values of $m=40$ and 160 (in both cases $\alpha=1$). For each of these $m$ values, four data sets are shown $n=100$, 200, 400 and 800. Excellent data collapse is obtained for large $n$ values for fixed $m$, and thus the scaling form (8) is very closely obeyed.

The reason why the scaling property with $n$ is preserved for any value of $m$ can be understood using a renormalization argument which will be discussed below. The system can be decomposed into a partition of elements, down to the level of $m=2$ elements. Each of these subsets will be described by the same functional dependence of the reduced variable $\delta$ and hence, the entire set has also to follow such a scaling. The justification of the invariance of the problem under renormalization is a question we will discuss in more detail in the next section.

Figure 6(*c*) also shows that the number of elements $m$ does influence significantly the shape of the function $F_m(\delta)$. In order to understand the $m$-dependence, it is instructive to study the time evolution of the process.

## 5. Time evolution

In the description of the model, we have not addressed the definition of 'elements' since they will be dependent on the type of problem the model is applied to. However, in many cases, we will be interested in considering the continuum limit of large $n$ and $M$. In such a case an 'element' will consist of many individual components. For this model to be of any relevance, one should be able to account for such an intermediate scale description. Indeed, we will show that the model can be renormalized to a coarse-grain description with similar rules.

In a system of $m$ elements, we imagine a partition in subsets of size $m'$. The difficulty is that the damage will have a statistical distribution in a subset, and thus it is

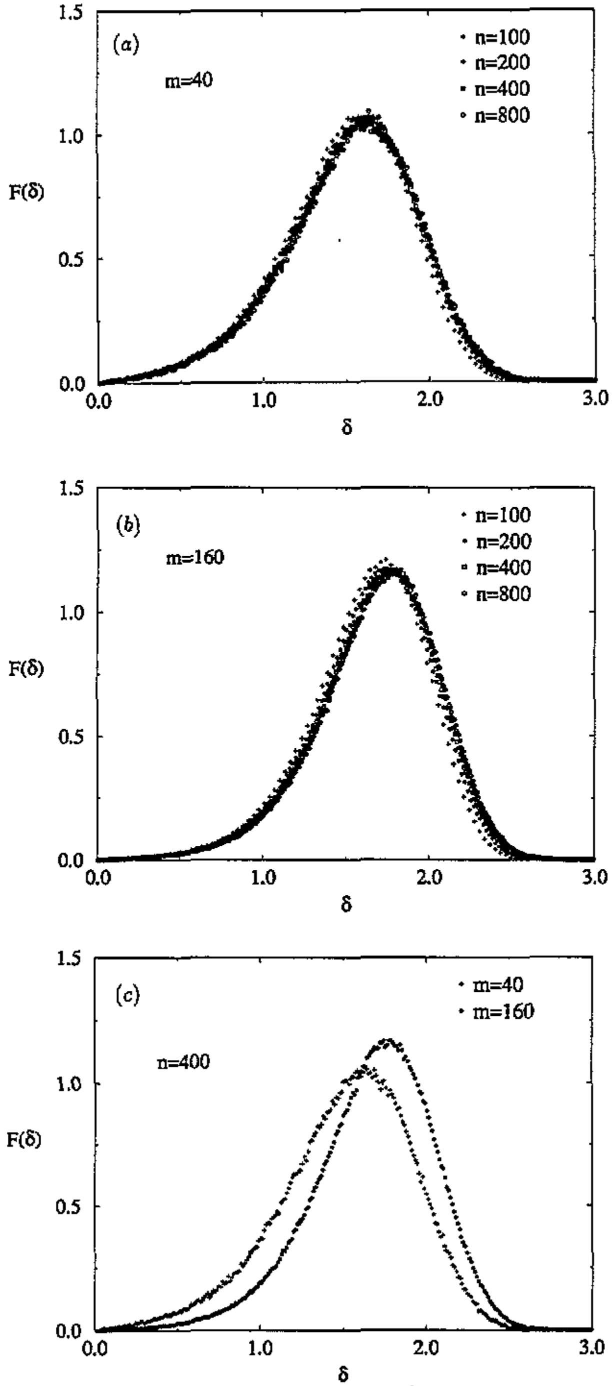


**Figure 6.** Two examples of data collapse for (*a*) $m = 40$ and (*b*) $m = 160$. The parameter $\alpha$ is one, and the exponent $\beta = 0.25$. Each distribution $F_m(\delta)$ is obtained by the superposition of four independent data sets corresponding to $n = 100$, 200, 400 and 800 (each being the result of 1000 different realizations). The reduced variables $F$ and $\delta$ capture very well the $n$-dependence, particularly for large $n$. (*c*) shows that a marked evolution of $F_m$ with $m$ is visible.

not possible to identify one single representative damage. However, it is not necessary to deal with the entire statistical distribution. Indeed, only three particular values of the damage are essential. Let us introduce the following *coarse-grained damages*:

• The 'time' has been counted proportionally to the total damage (i.e. $(\Sigma_i d_i)$. In order to estimate this 'time' if needed in a scale-independent fashion, we normalize the total damage by the system size, and thus we define the *average damage*:

$$d_{av} = \frac{1}{m'}\left(\sum_{i=1}^{m'} d_i\right). \tag{9}$$

This first coarse-grained damage is necessary to characterize the amount of damage that will be attributed to the subset at each time step.

• It is also necessary to characterize at any stage the probability of choosing a particular subset. The rules of the model are very well suited for this purpose. This probability is simply the sum of the probabilities of picking each component, and thus it is proportional to $(\Sigma_i \varphi(d_i))$. In order to keep the function $\varphi(d)$ identical for all levels of description, we introduce an *equivalent damage*, $d_{eq}$, so that $\varphi(d_{eq})$ is equal to the average value of $\varphi(d_i)$. Again, changing the sum into the average is innocuous because of the normalization of the probabilities. Thus the equivalent damage expression is

$$d_{eq} = \varphi^{(-1)}\left(\frac{1}{m'}\sum_{i=1}^{m'} \varphi(d_i)\right), \tag{10}$$

where $\varphi^{(-1)}$ is the inverse function of $\varphi$. Using this definition, the probability of damage to a subset is proportional to $\varphi(d_{eq})$ for this subset. Note that in the case where subsets have different sizes, this last expression has to be multiplied by the number of components of the concerned subset.

• Finally, in order to satisfy the stopping criterion, one needs to keep track of the most damaged site in the subset. Thus the *maximum damage* is the third and final coarse-grained damage we define:

$$d_{mx} = \max_{i=1}^{m'}(d_i). \tag{11}$$

We finally use the reduced form of the damage variable $\delta$ constructed so as to account for the $n$-dependence. The state of a subset is thus simply characterized by the three parameters $\delta_{av} = (1 - d_{av})n^{\beta}$, $\delta_{eq} = (1 - d_{eq})n^{\beta}$ and $\delta_{mx} = (1 - \delta_{mx})n^{\beta}$.

Let us now study how these parameters evolve for various values of $m$ and $n$. Figure 7 shows $\delta_{eq}$ as a function of $\delta_{av}$. We see that all points collapse onto a single curve for all $m$ and $n$. The physical meaning of this data collapse, is that it is possible to provide a scale-independent description of the *time-evolution* of the damage process.

Figure 8 shows that on the contrary, $\delta_{mx}$ is not a unique function of $\delta_{av}$, but that it depends significantly on $m$. The larger the system ($m$), the faster the maximum damage will increase. However a closer analysis reveals that the ratio $d_{eq}/d_{mx}$ is roughly constant apart from some initial transient regime, and that moreover, this ratio increases logarithmically with $m$ as shown on figure 9. Thus if the time evolution of the model can be easily coarse-grained, the *stopping criterion* has to be considered more carefully, since it is controlled by extreme statistics, and not by the most representative part of the distribution.

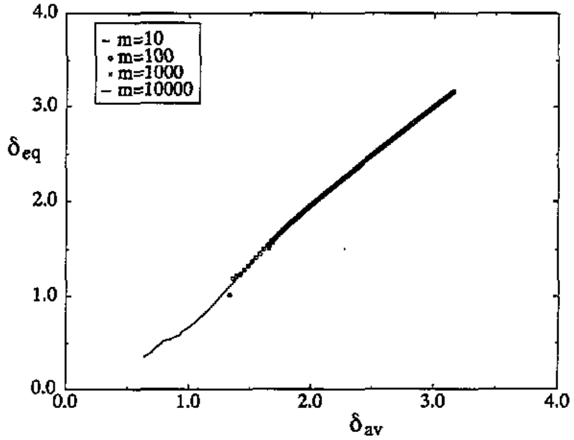


**Figure 7.** Relation between $\delta_{eq}$ and $\delta_{av}$ for different $m$ values indicated in the insert. The number of damage states was kept constant equal to $n = 100$. The $\alpha$ parameter is 1.

This can be understood as follows. As time proceeds, the distribution of damage can be seen as a very peaked function (representative of $d_{av}$ or $d_{eq}$) which is preceded by an exponential tail. The fact that $d_{eq}$ is independent of $m$ signifies that the weight of this exponential part is negligible compared to the peak where most elements are present. Therefore, we can consider the statistical distribution of damage indexed by a 'time', parameter, such as $t = d_{av}$, so that the probability that the damage of a site is in the interval $[d, d + \mathrm{d}d]$ is $C(d, d_{av})\,\mathrm{d}d$. Moreover, this distribution should be independent of $m$. Indeed, the probability to choose an element with a damage $d$ at time $d_{av}$ is $C(d, d_{av})\varphi(d)/\varphi(d_{eq})$. Since $d_{eq}$ is a function of $d_{av}$ which does not depend on $m$.

However, the largest damage will be such that

$$\int_{d_{mx}}^{1} C(d, d_{av})\,\mathrm{d}d = 1/m. \tag{12}$$

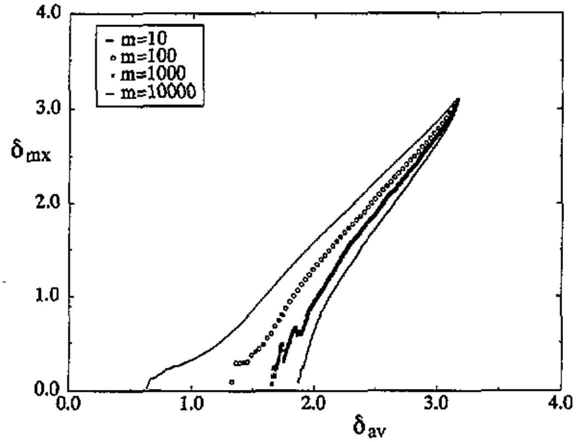


**Figure 8.** The maximum damage $\delta_{mx}$ is plotted as a function of the average damage $\delta_{av}$ for the same parameters as for figure 7. A systematic trend with $m$ is observed.

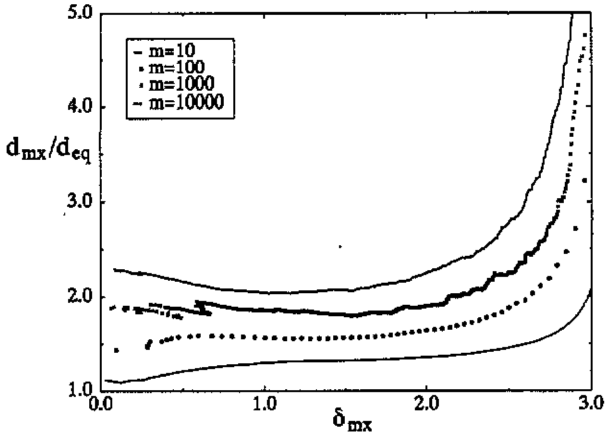


**Figure 9.** Ratio of the maximum damage to the equivalent damage $d_{mx}/d_{eq}$ as a function of the rescaled maximum damage $\delta_{mx}$. The same parameters as for figure 7 are used.

And thus $d_{mx}$ will be a function of $m$ as observed in figure 8. The assumption that the upper damage tail of the distribution is an exponential $C(d, d_{av}) \sim A \exp(-d/d_0)$ leads to a logarithmic dependence $d_{mx} \propto d_0 \log(m)$. Figure 9 shows in addition that $d_0$ is roughly proportional to $d_{av}$.

When $m$ increases, we have argued that the distribution $C(d, t)$ was not affected, but the large damage states with a low probability will be more densely occupied. Since the stopping criterion is a function of the maximum damage only $d_{mx} = 1$, the system will simply be arrested at a different time $d_{av} \propto 1/\log(m)$.

It is worth noting that $n$ and $m$ are pulling the system in opposite directions. At fixed $m$, when $n$ increases, the damage distribution becomes more narrow, and concentrated toward $d = 1$. On the contrary, when $m$ increases, at fixed $n$, the system is quenched at earlier stages of evolution. The weights of these two trends are very different (one is a power-law while the other is a logarithm). For instance, if both $n$ and $m$ increase equally, the $n$ effect will dominate.

## 6. Conclusion

We have introduced and discussed a statistical model to account for instabilities in collective systems. The number of damage states has been shown to lead to variations which could be accounted for by the introduction of the scaling variable $\delta = (1-d)n^\beta$. The value of $\beta$ has been related to the singularity of the weight function, $\varphi$. The number of elements shows an opposite trend which has been shown to be equivalent to stopping the evolution of the system at an earlier time.

Albeit the scaling behaviour observed in our model has some flavour of observed scaling laws for the applications we mentioned in the introduction, quantitative confrontations of the results with others requires the introduction of spatial correlations which are absent from the present mean-field version. In particular, the existence of a non-zero threshold damage appears to be a pathology of this mean-field approach, whereas the scaling with the number of damage states is similar to the one observed in the mentioned applications. Developments along those lines will be investigated in the future.

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